Consequences of De Giorgi-Nash-Moser

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1 Introduction

Here we will consider several consequences of the theorems discussed in the last several lectures. As before, we are interested in elliptic operators L given by

$$Lu = D_i(a^{ij} D_j u + b^i u) + c^j D_j u + du \text{ in } \Omega, \tag{1}$$

where Ω is a domain, $u \in W^{1,2}(\Omega)$, and the coefficients $a^{ij} \in L^\infty(\Omega)$, $b^i, c^i \in L^q(\Omega)$, and $d \in L^q(\Omega)$ satisfy

$$a^{ij}(x)\xi_i \xi_j \geq \lambda |\xi|^2 \text{ for a.e. } x \in \Omega \text{ and } \xi \in \mathbb{R}^n \tag{2}$$

for $\lambda > 0$ and

$$\sum_{i,j=1}^n |a^{ij}(x)|^2 \leq \Lambda^2 \text{ a.e. in } \Omega, \quad \lambda^{-2} \sum_{i,j=1}^n (\|b^i\|^2_{L^q(\Omega)} + \|c^i\|^2_{L^q(\Omega)}) + \lambda^{-1} \|d\|_{L^{q/2}(\Omega)} \leq \nu^2. \tag{3}$$

Recall we showed that:

Theorem 1. *Let L satisfy (2) and (3) and suppose $f^i \in L^q(\Omega)$ and $g \in L^{q/2}(\Omega)$ for $q > n$. Then if $u \in W^{1,2}(\Omega)$ satisfies $u \geq 0$ and*

$$Lu \geq D_i f^i + g \text{ weakly in } \Omega,$$

then for any ball $B_{2R}(x_0) \subset \Omega$ and $p > 1$,

$$\sup_{B_R(x_0)} u \leq C(R^{-n/p} \|u\|_{L^p(B_{2R}(x_0))} + \lambda^{-1} R^{1-n/q} \|f\|_{L^q(B_{2R}(x_0))} + \lambda^{-1} R^{2-2n/q} \|g\|_{L^{q/2}(B_{2R}(x_0))})$$

for $C = C(n, \Lambda/\lambda, \nu R, q, p) > 0$.

Theorem 2 (Weak Harnack inequality). *Let L satisfy (2) and (3) and suppose $f^i \in L^q(\Omega)$ and $g \in L^{q/2}(\Omega)$ for $q > n$. Then if $u \in W^{1,2}(\Omega)$ satisfies $u \geq 0$ and*

$$Lu \leq D_i f^i + g \text{ weakly in } \Omega,$$

then for any ball $B_{4R}(x_0) \subset \Omega$ and $1 \leq p < n/(n-2)$,

$$R^{-n/p} \|u\|_{L^p(B_{2R}(x_0))} \leq C(\inf_{B_R(x_0)} u + \lambda^{-1} R^{1-n/q} \|f\|_{L^q(B_{2R}(x_0)} + \lambda^{-1} R^{2-2n/q} \|g\|_{L^{q/2}(B_{2R}(x_0))})$$

for $C = C(n, \Lambda/\lambda, \nu R, q, p) > 0$.

Some important consequence of Theorems 1 and 2 are

1. Harnack inequality.
2. Strong maximum principle for equations in divergence form,
3. Theorem 1 holds without the assumption $u \geq 0$,
4. Estimate on the Hölder continuity of solutions.
5. Estimate on modulus of continuity at the boundary.

The Harnack inequality was discussed in the previous lectures. We will discuss the Hölder continuity estimates in lecture. For the other applications, we will state the theorems precisely below and the proofs may be covered in future example sheets.

2 Strong maximum principles

Theorem 3 (Strong maximum principle)**.** *Let Ω be an connected open set in $\mathbb{R}^n$. Let L be as in (1) such that (2) and (3) hold. Suppose $u \in W^{1,2}(\Omega)$ such that*

$$Lu \geq 0 \text{ weakly in } \Omega.$$

(a) If $b^i = d = 0$ a.e. in Ω, then there is no ball $B \subset\subset \Omega$ such that

$$\sup_B u = \sup_\Omega u \tag{4}$$

unless u is constant on Ω.

(b) If b^i and d satisfy

$$\int_\Omega (-b^i D_i \zeta + d\zeta) \leq 0 \tag{5}$$

for all $\zeta \in W_0^{1,1}(\Omega)$ with $\zeta \geq 0$ and $\sup_\Omega u \geq 0$, then there is no ball $B \subset\subset \Omega$ such that (4) holds unless u is constant on Ω.

(c) Without any restrictions on b^i and d, if $\sup_\Omega u = 0$, then there is no ball $B \subset\subset \Omega$ such that (4) holds unless u is constant on Ω.

Note that if $u \in W^{1,2}(\Omega)$, then it makes no sense to talk about u having an interior maximum at a point in Ω since u is only defined up to a set of Lebesgue measure zero. Thus instead we say u attains an interior maximum if (4) holds true. Of course, if u were also a continuous function, (4) is equivalent to u attaining its maximum value at a point in the interior of Ω.

3 Theorem 1 without $u \geq 0$

Theorem 4. *Let L be as in (1) such that (2) and (3) hold. Suppose $f^i \in L^q(\Omega)$ and $g \in L^{q/2}(\Omega)$ for $q > n$. Then if $u \in W^{1,2}(\Omega)$ with no sign restriction satisfies*

$$Lu \geq D_i f^i + g \text{ weakly in } \Omega,$$

then for any ball $B_{2R}(x_0) \subset \Omega$ and $p > 1$,

$$\sup_{B_R(x_0)} u \leq C(R^{-n/p}\|u\|_{L^p(B_{2R}(x_0))} + \lambda^{-1}R^{1-n/q}\|f\|_{L^q(B_{2R}(x_0))} + \lambda^{-1}R^{2-2n/q}\|g\|_{L^{q/2}(B_{2R}(x_0))})$$

for $C = C(n, \Lambda/\lambda, \nu R, q, p) \in (0, \infty)$.

4 Continuity estimates

Now we want to show that solutions $u \in W^{1,2}(\Omega)$ to a weak equation in divergence form is in fact in $C^{0,\mu}(\Omega)$ for some $\mu \in (0,1)$ with estimates on $[u]_{\mu,\Omega'}$, $\Omega' \subset\subset \Omega$. This is particularly important for the study of quasilinear elliptic equations.

Theorem 5. *Let L be as in (1) such that (2) and (3) hold. Suppose $f^i \in L^q(\Omega)$ and $g \in L^{q/2}(\Omega)$ for $q > n$. Then if $u \in W^{1,2}(\Omega)$ satisfies*

$$Lu = D_i f^i + g \text{ weakly in } \Omega,$$

then for any ball $B_{R_0}(x_0) \subset \Omega$ and $R \leq R_0$,

$$\operatorname{osc}_{B_R(x_0)} u \leq C\left(\frac{R}{R_0}\right)^{\mu}\left(\operatorname{osc}_{B_{R_0}(x_0)} u + \frac{C}{\lambda}(R_0^{1-n/q}\|f\|_{L^q(\Omega)} + R_0^{2-2n/q}\|g\|_{L^{q/2}(\Omega)})\right) \tag{6}$$

for $\mu \in (0,1)$ and $C \in (0,\infty)$ depending on n, Λ/λ, νR_0, and q, where

$$\operatorname{osc}_{B_R(x_0)} u = \sup_{B_R(x_0)} u - \inf_{B_R(x_0)} u$$

Proof. We may assume WLOG that $R \leq R_0/4$. Let $M_0 = \sup_{B_{R_0}(x_0)} |u|$, $m_1 = \inf_{B_R(x_0)} u$, $M_1 = \sup_{B_R(x_0)} u$, $m_4 = \inf_{B_{4R}(x_0)} u$, and $M_4 = \sup_{B_{4R}(x_0)} u$. Then $M_4 - u$ and $u - m_4$ are nonnegative functions on $B_{4R}(x_0)$ satisfying

$$\begin{aligned} L(M_4 - u) &= M_4(D_i b^i + d) - D_i f^i - g, \\ L(u - m_4) &= -m_4(D_i b^i + d) + D_i f^i + g. \end{aligned}$$

Let

$$\begin{aligned} k(R) = {} & \lambda^{-1}R^{1-n/q}(\|f\|_{L^q(B_{R_0}(x_0))} + M_0\|b\|_{L^q(B_{R_0}(x_0))}) \\ & + \lambda^{-1}R^{2-2n/q}(\|g\|_{L^{q/2}(B_{R_0}(x_0))} + M_0\|d\|_{L^{q/2}(B_{R_0}(x_0))}) \end{aligned}$$

By the weak Harnack inequality applied to $M_4 - u$ and $u - m_4$ with $p = 1$,

$$R^{-n} \int_{B_{2R}(x_0)} (M_4 - u) \le C(M_4 - M_1 + k(R)),$$
$$R^{-n} \int_{B_{2R}(x_0)} (u - m_4) \le C(m_1 - m_4 + k(R)),$$

for some constant $C \in (1, \infty)$. By addition,

$$M_4 - m_4 \le C(M_4 - m_4 - M_1 + m_1 + k(R));$$

that is

$$\operatorname{osc}_{B_R(x_0)} u \le \gamma \operatorname{osc}_{B_{4R}(x_0)} u + k(R) \tag{7}$$

where $\gamma = 1 - 1/C \in (0, 1)$ depends on n, Λ/λ, νR_0, and q.

Observe that (7) implies that as we decrease the radius of a ball by a factor of 4, the the oscillation of u decays by a small factor of γ. Inequalities such as this are common in differential equations and geometric analysis. The standard thing to do now is to iterate the inequality (7), which in this case will establish (6). Let $R_1 \le R_0$. Iterating (7) with $R = 4^{-m} R_1$ for $m = 1, 2, 3, \ldots$ we obtain

$$\begin{aligned}\operatorname{osc}_{B_{4^{-m}R_1}(x_0)} u &\le \gamma^m \operatorname{osc}_{B_{R_1}(x_0)} u + \sum_{i=0}^{m-1} \gamma^i k(R_1) \\ &\le \gamma^m \operatorname{osc}_{B_{R_1}(x_0)} u + \frac{1}{1-\gamma} k(R_1).\end{aligned}$$

Now consider any radius $R \le R_1$ and choose an integer $m \ge 1$ such that $4^{-m-1} R_1 < R \le 4^{-m} R_1$ to get

$$\begin{aligned}\operatorname{osc}_{B_R(x_0)} u &\le \operatorname{osc}_{B_{4^{-m}R_1}(x_0)} u \\ &\le \gamma^m \operatorname{osc}_{B_{R_1}(x_0)} u + \frac{1}{1-\gamma} k(R_1) \\ &\le \frac{1}{\gamma} \left(\frac{R}{R_1}\right)^{-\log\gamma/\log 4} \operatorname{osc}_{B_{R_0}(x_0)} u + \frac{1}{1-\gamma} k(R_1).\end{aligned}$$

Let $R_1 = R_0^{1-\tau} R^{\tau}$ for $\tau \in (0, 1)$. Then

$$\begin{aligned}\operatorname{osc}_{B_R(x_0)} u &\le \frac{1}{\gamma} \left(\frac{R}{R_0}\right)^{-(1-\tau)\log\gamma/\log 4} \operatorname{osc}_{B_{R_0}(x_0)} u + \frac{1}{1-\gamma} k(R_0^{1-\tau} R^{\tau}) \\ &\le \frac{1}{\gamma} \left(\frac{R}{R_0}\right)^{-(1-\tau)\log\gamma/\log 4} \operatorname{osc}_{B_{R_0}(x_0)} u + \frac{1}{1-\gamma} \left(\frac{R}{R_0}\right)^{\tau(1-n/q)} k(R_0).\end{aligned}$$

Choose τ such that $-(1-\tau)\log\gamma/\log 4 < \tau(1 - n/q)$ and choose $\mu = -(1-\tau)\log\gamma/\log 4 \in (0, 1)$. □

As an immediate consequence of the above, we obtain:

Corollary 1. *Let L be as in (1) such that (2) and (3) hold. Suppose $f^i \in L^q(\Omega)$ and $g \in L^{q/2}(\Omega)$ for $q > n$. Then if $u \in W^{1,2}(\Omega)$ satisfies*

$$Lu = D_i f^i + g \text{ weakly in } \Omega,$$

then $u \in C^{0,\mu}(\Omega)$ and for any $B_{R_0}(x_0) \subset\subset \Omega$,

$$R_0^\mu [u]_{\mu, B_{R_0/4}(x_0)} \leq C(\sup_{B_{R_0}(x_0)} |u| + \lambda^{-1}(R_0^{1-n/q}\|f\|_{L^q(\Omega)} + R_0^{2-2n/q}\|g\|_{L^{q/2}(\Omega)}))$$

for $\mu \in (0,1)$ and $C \in (0,\infty)$ depending on n, Λ/λ, νR_0, q, and $d = \operatorname{dist}(\Omega', \partial\Omega)$.

Proof. Let $x, y \in B_{R_0/4}(x_0)$ and let $R = |x-y|$. Then

$$\begin{aligned}
\frac{|u(x)-u(y)|}{|x-y|^\mu} &\leq R^{-\mu} \operatorname{osc}_{B_R(x)} u \\
&\leq CR_0^{-\mu}(\operatorname{osc}_{B_{R_0}(x_0)} u + \lambda^{-1}(R_0^{1-n/q}\|f\|_{L^q(\Omega)} + R_0^{2-2n/q}\|g\|_{L^{q/2}(\Omega)})) \\
&\leq CR_0^{-\mu}(\sup_{B_{R_0}(x_0)} |u| + \lambda^{-1}(R_0^{1-n/q}\|f\|_{L^q(\Omega)} + R_0^{2-2n/q}\|g\|_{L^{q/2}(\Omega)})).
\end{aligned}$$

□

Note that by the interpolation inequality

$$\sup_{B_{R_0}(x_0)} |u| \leq C\|u\|_{L^2(B_{R_0}(x_0))} + \varepsilon[u]_{\mu, B_{R_0}(x_0)}$$

for $C = C(\varepsilon, n, \mu) \in (0,\infty)$ (left as an exercise), we obtain

$$R_0^\mu [u]_{\mu, B_{R_0/4}(x_0)} \leq C(\|u\|_{L^2(B_{R_0}(x_0))} + \lambda^{-1}(R_0^{1-n/q}\|f\|_{L^q(\Omega)} + R_0^{2-2n/q}\|g\|_{L^{q/2}(\Omega)})) \tag{8}$$

for some constant $C \in (0,\infty)$ depending on n, Λ/λ, νR_0, q, and $d = \operatorname{dist}(\Omega', \partial\Omega)$. Alternatively, (8) follows from Theorem 1. Given $\Omega' \subset\subset \Omega$, we can cover Ω' by balls $B_{R_j/4}(x_j)$ for $x_j \in \Omega$ and $R_j < d = \operatorname{dist}(\Omega', \partial\Omega)$ to get

$$[u]_{\mu,\Omega'} \leq C(\|u\|_{L^2(\Omega)} + \lambda^{-1}(\|f\|_{L^q(\Omega)} + \|g\|_{L^{q/2}(\Omega)})).$$

for $\mu \in (0,1)$ and $C \in (0,\infty)$ depending on n, Λ/λ, νR_0, q, and $d = \operatorname{dist}(\Omega', \partial\Omega)$.

5 Continuity estimates at the boundary

Finally we want to obtain continuity estimates at the boundary of Ω. First we have the following:

Theorem 6. *Let L satisfy (2) and (3) and suppose $f^i \in L^q(\Omega)$ and $g \in L^{q/2}(\Omega)$ for $q > n$. Then if $u \in W^{1,2}(\Omega)$ satisfies $u \geq 0$ in $\Omega \cap B_{4R}(x_0)$ and*

$$Lu \leq D_i f^i + g \text{ weakly in } \Omega,$$

then for any ball $B_{4R}(x_0) \subset \mathbb{R}^n$ and $1 \leq p < n/(n-2)$,

$$R^{-n/p}\|u_m^-\|_{L^p(B_{2R}(x_0))} \leq C(\inf_{B_R(x_0)} u_m^- + \lambda^{-1}R^{1-n/q}\|f\|_{L^q(B_{2R}(x_0)} + \lambda^{-1}R^{2-2n/q}\|g\|_{L^{q/2}(B_{2R}(x_0))})$$

for $C = C(n, \Lambda/\lambda, \nu R, q, p) > 0$, where $m = \inf_{\partial\Omega \cap B_{4R}(x_0)} u$ and

$$u_m^-(x) = \begin{cases} \min\{u(x), m\} & \text{if } x \in \Omega, \\ m & \text{if } x \notin \Omega. \end{cases}$$

Proof. Modify the proof of the weak Harnack inequality, letting $\bar{u} = u_m^- + k$ and using the test function $\zeta = \eta^2(\bar{u}^\beta - (M+k)^\beta)$ for $M = \sup_{\partial\Omega \cap B_{2R}(x_0)} u^+$ if $\beta > 0$ and $\zeta = \eta^2(\bar{u}^\beta - (m+k)^\beta)$ if $\beta < 0$ and noting that $\zeta \leq \eta^2 \bar{u}^\beta$. □

Theorem 7. *Suppose L be as in (1) such that (2) and (3) hold. Suppose $f^i \in L^q(\Omega)$ and $g \in L^{q/2}(\Omega)$ for $q > n$. Suppose $u \in W^{1,2}(\Omega)$ satisfies*

$$Lu = D_i f^i + g \text{ weakly in } \Omega.$$

Finally, suppose Ω satisfies the exterior cone condition at $\xi \in \partial\Omega$, i.e. there exists a finite right circular cone V_ξ with vertex ξ such that $\overline{\Omega} \cap V_\xi = \{\xi\}$ (for example, if Ω is a $C^{0,1}$ domain). Then for any $0 < R \leq R_0$,

$$\operatorname{osc}_{\Omega \cap B_R(\xi)} u \leq C \left(\frac{R}{R_0}\right)^\mu \left(\sup_{B_{R_0}(\xi)} |u| + \frac{C}{\lambda}(R_0^{1-n/q} \|f\|_{L^q(\Omega)} + R_0^{2-2n/q} \|g\|_{L^{q/2}(\Omega)})\right) + \operatorname{osc}_{\partial\Omega \cap B_{\sqrt{RR_0}}(\xi)} u. \tag{9}$$

for $\mu \in (0,1)$ and $C > 0$ depending on n, Λ/λ, νR_0, q, and V_ξ.

Proof. We follow the proof of Theorem 1. We may assume WLOG that R is less than or equal to both $R_0/4$ and the height of V_ξ. Let $M_0 = \sup_{\Omega \cap B_{R_0}(\xi)} |u|$, $m_1 = \inf_{\Omega \cap B_R(\xi)} u$, $M_1 = \sup_{\Omega \cap B_R(\xi)} u$, $m_4 = \inf_{\Omega \cap B_{4R}(\xi)} u$, and $M_4 = \sup_{\Omega \cap B_{4R}(\xi)} u$. By Theorem 6 applied to $M_4 - u$ and $u - m_4$ with $p = 1$,

$$\begin{aligned}
(M_4 - M) \frac{|B_{2R}(\xi) \setminus \Omega|}{R^n} &\leq R^{-n} \int_{B_{2R}(\xi)} (M_4 - u)^-_{M_4 - M} \leq C(M_4 - M_1 + k(R)), \\
(m - m_4) \frac{|B_{2R}(\xi) \setminus \Omega|}{R^n} &\leq R^{-n} \int_{B_{2R}(\xi)} (u - m_4)^-_{m - m_4} \leq C(m_1 - m_4 + k(R)),
\end{aligned}$$

for some constant $C \in (1, \infty)$. By the exterior cone condition,

$$\begin{aligned}
(M_4 - M) &\leq C(M_4 - M_1 + k(R)), \\
(m - m_4) &\leq C(m_1 - m_4 + k(R)).
\end{aligned}$$

By addition,

$$M_4 - m_4 - M + m \leq C(M_4 - m_4 - M_1 + m_1 + k(R)).$$

that is

$$\operatorname{osc}_{B_R(x_0)} u \leq \gamma \operatorname{osc}_{B_{4R}(x_0)} u + k(R) + \operatorname{osc}_{\partial\Omega \cap B_R(\xi)} u$$

where $\gamma = 1 - 1/C \in (0,1)$ depends on n, Λ/λ, νR_0, and q. (9) follows. □

Corollary 2. *Let Ω be a domain in $\mathbb{R}^n$ that satisfies the exterior cone condition at each $\xi \in \partial\Omega$. Let L be as in (1) such that (2) and (3) hold. Suppose $f^i \in L^q(\Omega)$ and $g \in L^{q/2}(\Omega)$ for $q > n$ and $\varphi \in W^{1,2}(\Omega) \cap C^0(\overline{\Omega})$. Suppose $u \in W^{1,2}(\Omega)$ satisfies*

$$\begin{aligned}
Lu &= D_i f^i + g \text{ weakly in } \Omega, \\
u &= \varphi \text{ on } \partial\Omega.
\end{aligned}$$

Then $u \in C^0(\overline{\Omega})$ with $u = \varphi$ pointwise on Ω.

Corollary 3. *Let Ω be a domain in $\mathbb{R}^n$ that satisfies the exterior cone condition at each $\xi \in \partial\Omega$. Let L be as in (1) such that (2), (3), and (5) hold. Suppose $f^i \in L^q(\Omega)$ and $g \in L^{q/2}(\Omega)$ for $q > n$ and $\varphi \in C^0(\partial\Omega)$. Then there is a unique $u \in W^{1,2}(\Omega) \cap C^0(\overline{\Omega})$ such that*

$$\begin{aligned} Lu &= D_i f^i + g \text{ weakly in } \Omega, \\ u &= \varphi \text{ pointwise on } \partial\Omega. \end{aligned}$$